

Strain Transformation and Rosette Gage Theory

It is often desired to measure the full state of strain on the surface of a part, that is to measure not only the two extensional strains, ϵ_x and ϵ_y , but also the shear strain, γ_{xy} , with respect to some given xy axis system. It should be clear from the previous discussion of the electrical resistance strain gage that a single gage is capable only of measuring the extensional strain in the direction that the gage is oriented. Assuming that the x and y axes are specified, it would be possible to mount two gages in the x and y directions, respectively to measure the associated extensional strains in these directions. However, there is no direct way to measure the shear strain, γ_{xy} . Nor is it possible to directly measure the principal strains since the principal directions are not generally known.

The solution to this problem lies in recognizing that the 2D state of strain at a point (on a surface) is defined by three independent quantities which can be taken as either: (a) ϵ_x , ϵ_y , and γ_{xy} , or (b) ϵ_1 , ϵ_2 , and θ , where case (a) refers to strain components with respect to an arbitrary xy axis system, and case (b) refers to the principal strains and their directions. Either case fully defines the state of 2D strain on the surface and can be used to compute strains with respect to any other coordinate system. This situation implies that it should be possible to determine these 3 independent quantities if it is possible to make three independent measurements of strain at a point on the surface. The most obvious approach is to place three strain gages together in a “rosette” with each gage oriented in a different direction and with all of them located as close together as possible to approximate a measurement at a point. As will be shown below, if the three strains and the gage directions are known, it is possible to solve for the principal strains and their directions or equivalently, the state of strain with respect to an arbitrary xy coordinate system. The relations needed are the strain transformation equations and Mohr’s Circle construction provides a good visualization of this process.

2D Strain Transformation and Mohr’s Circle

The two dimensional strain transformation equations were developed in earlier structural mechanics courses and are very similar to the 2D stress transformation equations (see for example, chapter 6 in, Gere & Timoshenko, Mechanics of Materials, 3rd edition, 1990). Without rederivation, they are repeated below:

$$\begin{aligned}\epsilon_{x'} &= \epsilon_x \cos^2 \theta + \epsilon_y \sin^2 \theta + \gamma_{xy} \sin \theta \cos \theta \\ \epsilon_{y'} &= \epsilon_x \sin^2 \theta + \epsilon_y \cos^2 \theta - \gamma_{xy} \sin \theta \cos \theta \\ \gamma_{x'y'} &= 2(\epsilon_y - \epsilon_x) \sin \theta \cos \theta + \gamma_{xy} (\cos^2 \theta - \sin^2 \theta)\end{aligned}\tag{1}$$

These transformation equations involve squares and products of sine and cosine functions and these can be replaced with double-angle results to yield the double-angle form of the transformation equations:

$$\begin{aligned}
\varepsilon_{x'} &= \frac{1}{2}(\varepsilon_x + \varepsilon_y) + \frac{1}{2}(\varepsilon_x - \varepsilon_y) \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta \\
\varepsilon_{y'} &= \frac{1}{2}(\varepsilon_x + \varepsilon_y) - \frac{1}{2}(\varepsilon_x - \varepsilon_y) \cos 2\theta - \frac{\gamma_{xy}}{2} \sin 2\theta \\
\gamma_{x'y'} &= -(\varepsilon_x - \varepsilon_y) \sin 2\theta + \gamma_{xy} \cos 2\theta
\end{aligned}
\tag{2}$$

Given the ready availability of powerful calculators and spreadsheets, evaluation of these transformation equations is a relatively simple matter today and involves little more than a few seconds to enter the formula in a calculator or spreadsheet cell. This was not always so simple a task and the Mohr's Circle graphical representation was developed long ago to aid in this process. Today, Mohr's Circle is not needed for graphical calculation, but it does provide a good visualization of the transformation equations and the geometry can be used to infer the actual form for the equations needed for execution in a calculator or computer.

The double-angle form of the transformation relations involve simple sine and cosine terms along with a constant that should suggest equations of a circle with center away from the coordinate origin. The trick here is to identify the appropriate new "x" and "y" values to plot to construct a circle. This is perhaps easier to do by explaining the Mohr's Circle than it is to actually deduce the form directly. Figure 1 below shows Mohr's Circle for a state of strain defined by an xy axis. Assume for the moment that the coordinates of the opposite ends (X-Y) of the indicated diameter of the circle define the strain state, ε_x , ε_y , and γ_{xy} , in the xy axis system. Note that both the ε_x and ε_y extensional strains are plotted on the abscissa (x axis) while one-half the shear strain, $\gamma_{xy}/2$, is plotted on the ordinate (y axis). The positive direction for γ_{xy} , is taken to be downward (consistent with Gere & Timoshenko). One can construct the circle by first plotting the $(\varepsilon_x, \gamma_{xy}/2)$ pair as point X on the diameter. Next, the pair $(\varepsilon_y, -\gamma_{xy}/2)$ are plotted as the opposite end of the diameter, Y, and the circle can be constructed with X-Y as the diameter. This is the basic Mohr's Circle and it always has its center on the abscissa at a point given by the value, $(\varepsilon_x + \varepsilon_y)/2$. The circle diameter is easily computed as: $D = \sqrt{[\gamma_{xy}]^2 + (\varepsilon_x - \varepsilon_y)^2}$.

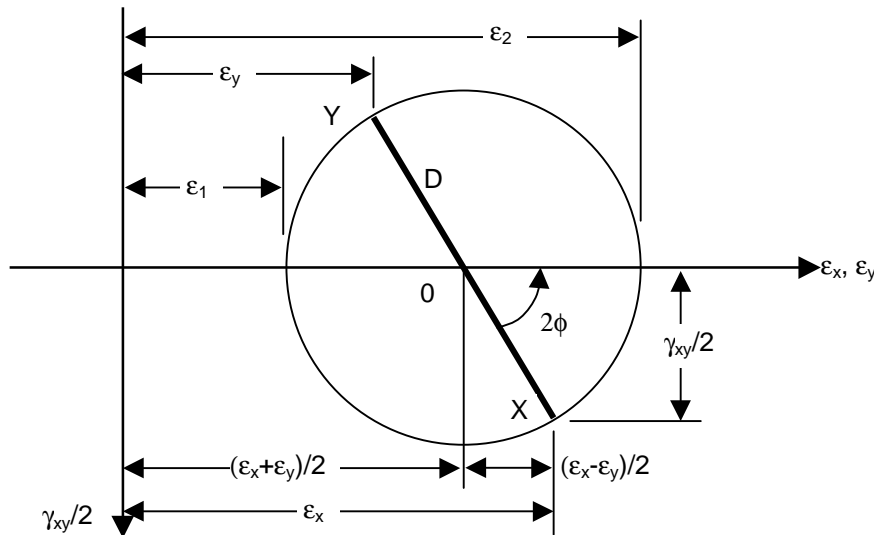


Figure 1. Basic Mohr's Circle Geometry

So far, there is no clear connection to the double-angle transformation equations above, but this will become evident in a moment! To calculate a new state of strain, $\epsilon_{x'}$, $\epsilon_{y'}$ and $\gamma_{x'y'}$ in an $x'y'$ axis system rotated θ counterclockwise from the xy axis system, one must construct a new diameter for Mohr's Circle rotated 2θ counterclockwise from the initial X - Y diameter as shown below in Figure 2. The coordinates of the new diameter endpoints, X' - Y' , represent the new strain state as computed from the double-angle strain transformation equations. This should be clear by inspection of the Mohr's Circle and by evaluation of the circle geometry as shown. Note that rotation in Mohr's Circle is always $2x$ the geometric rotation so that the state of strain defined by opposite ends of a diameter of the circle (e.g., 180° apart) correspond to strains that are at $180^\circ/2=90^\circ$ in geometric axes.

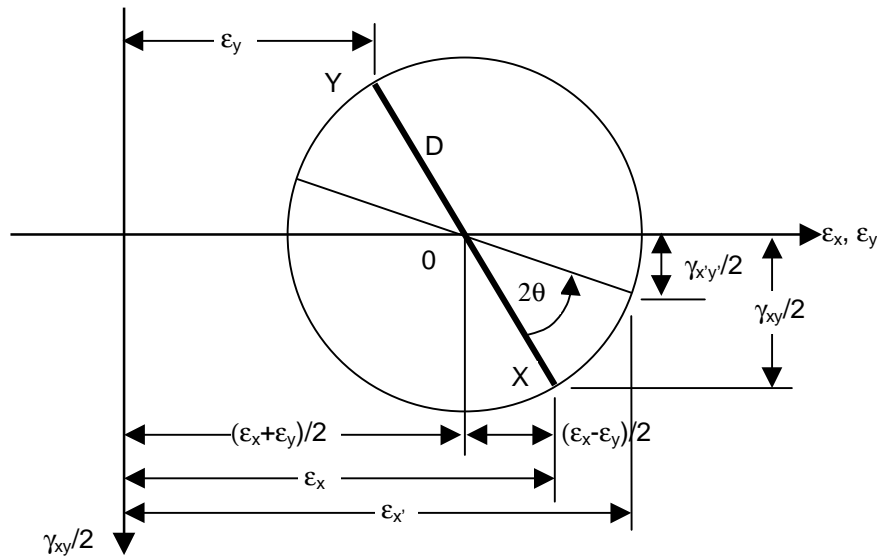


Figure 2. Mohr's Circle for Strain Transformation of θ

It should be pointed out also that this particular form for Mohr's Circle uses an inverted ordinate (y axis) and positive θ is counterclockwise. Another popular form for Mohr's Circle uses an upward positive ordinate axis and positive clockwise θ .

It is strikingly apparent from Mohr's Circle in Figure 1 that there is a state of strain that does not involve any shear strain and this corresponds to the circle diameter lying along the abscissa and defined by the two points, ϵ_1 and ϵ_2 as shown. These are called the principal strains and they are associated with an $x'y'$ axis system rotated, ϕ , counterclockwise from the xy axis (2ϕ in Mohr's Circle). This defines not only the principal strains but also the axes on which they act. Figure 3 shows other useful Mohr's Circle configurations.

Strain Gage Rosettes

Strain gage rosettes consist of two or more co-located strain gages oriented at a fixed angle with respect to each other. Strict co-location of the gages requires mounting each individual gage on top of the others in what is called a "stacked" rosette, but this leads to a complicated and often inaccurate type of gage. The more common approach is to place the gages in a tightly packed pattern as close as possible to the rosette center. Rosettes typically involve 2, 3 or 4

strain gages with relative orientations of 30° , 45° , 60° or 90° . Figure 4 shows several examples. At least 3 independent strain readings are needed to define the 2D state of strain if no other information is available so the 3-gage rosettes are the most popular (the 90° 2-gage rosette can be used to measure principal strains when the principal direction is known and the gage can be oriented accordingly). The Rectangular Rosette and the Delta Rosette are the most commonly used 3-gage rosettes because of their simple geometry.

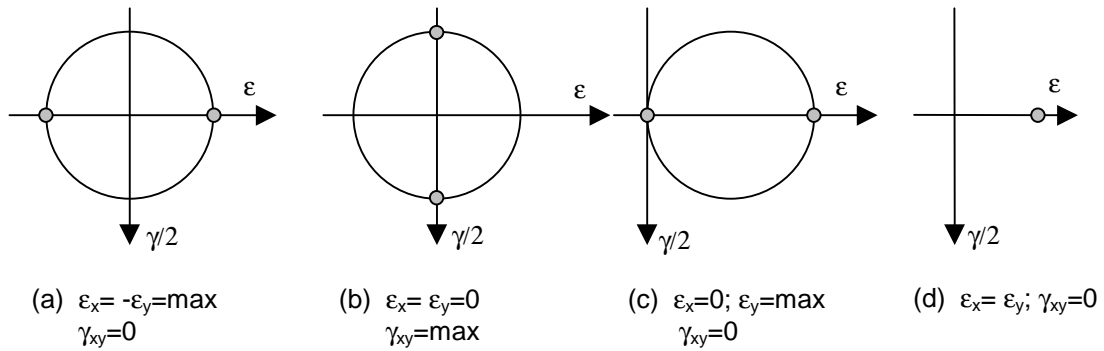


Figure 3. Some Useful Mohr's Circle Configurations

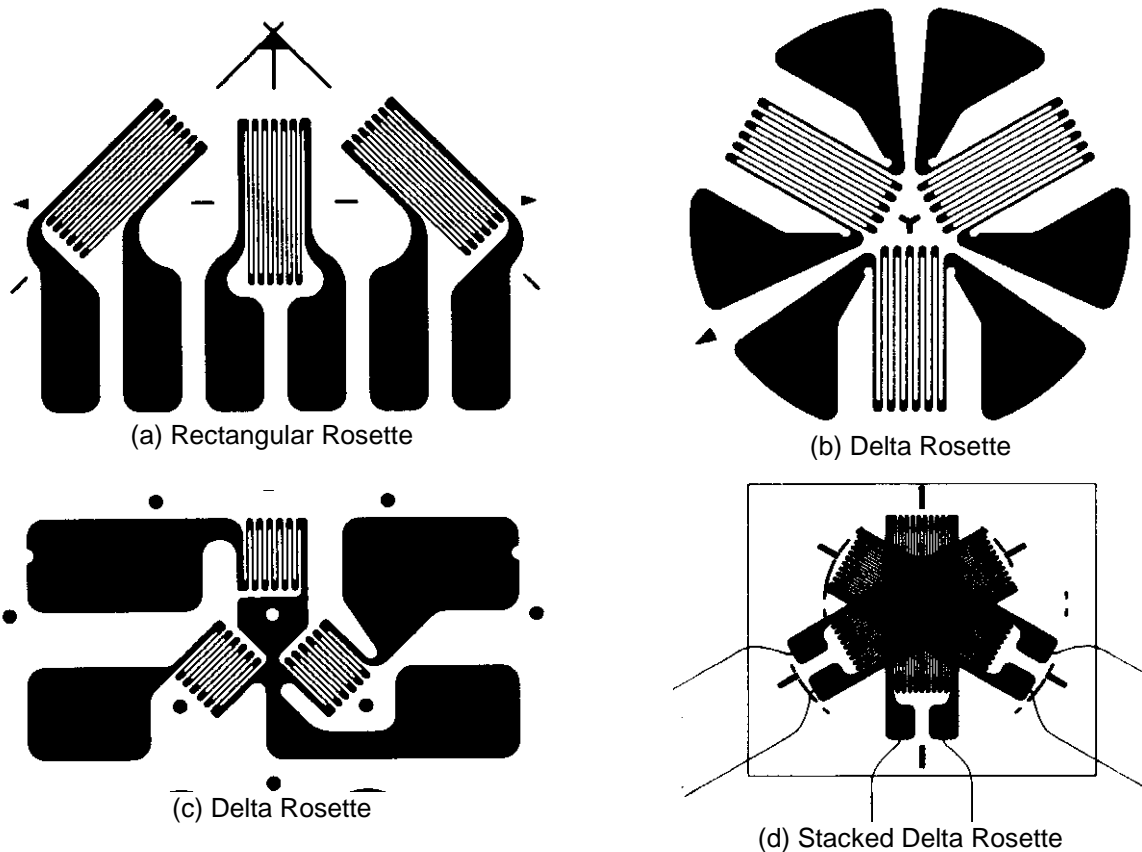


Figure 4. Typical Strain Gage Rosettes (5x scale)

Rectangular Rosette Gage Equations

Given the measurement of 3 independent strains from the 3 gages in a rectangular rosette it is possible to calculate the principal strains and their orientation with respect to the rosette gage. It is also readily possible to calculate the state of strain at the gage location with respect to any particular xy axis system using either the rosette readings or the principal strains and their axis orientation. To illustrate this, consider a situation in which the rosette is oriented with gages labeled A, B and C at 45° apart as shown in Figure 5. Assume also that the principal strains at the rosette are oriented at an angle, ϕ , to the rosette gage A axis. For this case, it is easy to use the strain transformation equations (Eq. 1) to calculate the strain in each rosette gage in terms of the principal strains and the angle, ϕ , (simply assume $\epsilon_x = \epsilon_1$ and $\epsilon_y = \epsilon_2$ and compute ϵ_x' for angles of rotation ϕ , $\phi + 45^\circ$, and $\phi + 90^\circ$) to yield three equations:

$$\epsilon_A = \frac{\epsilon_1 + \epsilon_2}{2} + \frac{\epsilon_1 - \epsilon_2}{2} \cos 2\phi \quad (3a)$$

$$\epsilon_B = \frac{\epsilon_1 + \epsilon_2}{2} + \frac{\epsilon_1 - \epsilon_2}{2} \cos 2(\phi + 45^\circ) \quad (3b)$$

$$\epsilon_C = \frac{\epsilon_1 + \epsilon_2}{2} + \frac{\epsilon_1 - \epsilon_2}{2} \cos 2(\phi + 90^\circ) \quad (3c)$$

These are 3 simultaneous equations relating ϵ_A , ϵ_B , ϵ_C to ϵ_1 , ϵ_2 , ϕ . It is a relatively simple matter to invert the equations and solve for ϵ_1 , ϵ_2 , ϕ in terms of ϵ_A , ϵ_B , ϵ_C yielding:

$$\epsilon_{1,2} = \frac{\epsilon_A + \epsilon_C}{2} \pm \frac{1}{\sqrt{2}} \sqrt{(\epsilon_A - \epsilon_B)^2 + (\epsilon_B - \epsilon_C)^2} \quad (4a,b)$$

$$\phi = \frac{1}{2} \tan^{-1} \left(\frac{\epsilon_A - 2\epsilon_B + \epsilon_C}{\epsilon_A - \epsilon_C} \right) \quad (4c)$$

Eq. 4 can be used to compute the principal strains and the principal axis orientation directly from the rectangular rosette gage readings. *Note that there are many different possible gage numbering arrangements besides the particular A,B,C layout here, and they can lead to forms for the final results shown above but with A, B and C interchanged.)*

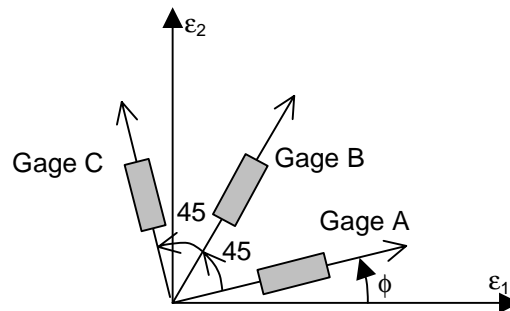


Figure 5. Rectangular Rosette Gage Orientation

It should be noted that the above results can also be developed directly from the Mohr's Circle representation with about the same amount of effort and perhaps a bit more visualization of the results. It is somewhat simpler to assume an x-y axis system with the rosette oriented such that gage A is along the x axis and gage C is along the y axis. From Eq. 3 it follows that:

$$\epsilon_A = \frac{1}{2}(\epsilon_x + \epsilon_y) + \frac{1}{2}(\epsilon_x - \epsilon_y) \cos(2 \times 0^\circ) + \frac{1}{2} \gamma_{xy} \sin(2 \times 0^\circ) = \epsilon_x$$

$$\epsilon_B = \frac{1}{2}(\epsilon_x + \epsilon_y) + \frac{1}{2}(\epsilon_x - \epsilon_y) \cos(2 \times 45^\circ) + \frac{1}{2} \gamma_{xy} \sin(2 \times 45^\circ) = \frac{1}{2}(\epsilon_x + \epsilon_y) + \frac{1}{2} \gamma_{xy}$$

$$\epsilon_C = \frac{1}{2}(\epsilon_x + \epsilon_y) + \frac{1}{2}(\epsilon_x - \epsilon_y) \cos(2 \times 90^\circ) + \frac{1}{2} \gamma_{xy} \sin(2 \times 90^\circ) = \epsilon_y$$

These equations can readily be solved for the strains in the x-y system:

$$\epsilon_x = \epsilon_A$$

$$\epsilon_y = \epsilon_C$$

$$\gamma_{xy} = 2\epsilon_B - \epsilon_A - \epsilon_C$$

This defines the strain state at the rosette with respect to the assumed x-y axis system. It is a simple matter to now construct a Mohr's Circle and from this to compute the principal strains and their orientation with respect to the x-y axis (and therefore the rosette). Figure 6 summarizes these results: It is a simple matter to compute ϵ_1 and ϵ_2 from the Mohr's Circle center and radius given in Figure 6, and the results are identical to Eq. 4 above. Figure 6 also reveals clearly that the maximum shear strain is given by 2 times the radius of the Mohr's Circle, and in this case it can be computed in terms of principal strains as: $\gamma_{\max} = |(\epsilon_1 - \epsilon_2)|/2$.

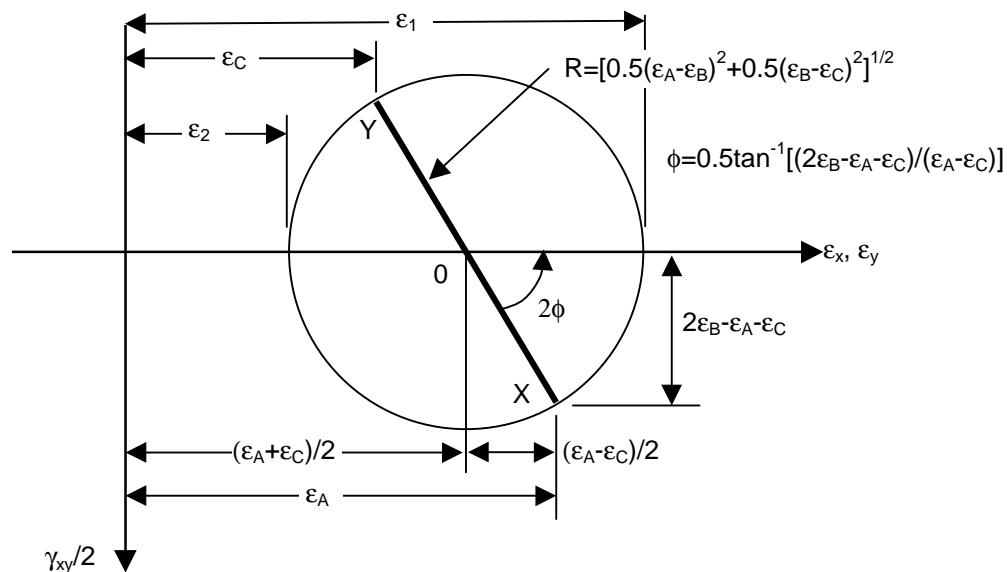


Figure 6. Mohr's Circle for a Rectangular Rosette

Delta Rosette

The Delta Rosette consists of the 3 strain gages oriented at 120° to each other as legs of an equilateral triangle. As for the Rectangular Rosette, Figure 7 shows the gage orientation but this time with gages A, B and C 60° apart. (Note that although it is shown at 240° from gage A, the

gage B axis also bisects the gage A and gage C axes and can therefore also be considered at making an angle of 60° to A and C. As before, it is simply a matter of applying the strain transformation equation from the principal axes to gage A at ϕ , gage B at $\phi+60^\circ$ and gage C at $\phi+120^\circ$, yielding 3 equations in 3 unknowns.

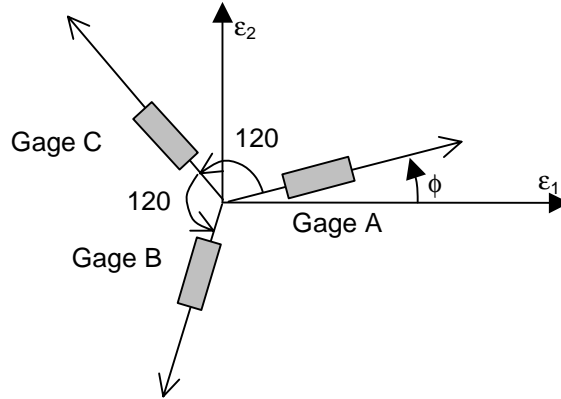


Figure 7. Delta Rosette Gage Orientation

Following a similar approach to that employed for the analysis of the Rectangular Rosette, it is possible to show that the principal strains are given by:

$$\varepsilon_{1,2} = \frac{\varepsilon_A + \varepsilon_B + \varepsilon_C}{3} \pm \frac{\sqrt{2}}{3} \sqrt{(\varepsilon_A - \varepsilon_B)^2 + (\varepsilon_B - \varepsilon_C)^2 + (\varepsilon_C - \varepsilon_A)^2}$$

$$\phi = \frac{1}{2} \tan^{-1} \left[\frac{\sqrt{3}(\varepsilon_C - \varepsilon_B)}{2\varepsilon_A - \varepsilon_B - \varepsilon_C} \right]$$
(5)

Principal Stresses

It should be pointed out that the above results involve strain only and do not describe the state of stress at the rosette. In order to determine the stress state, it is necessary to use the stress-strain relations to express the stress components in terms of the strain components. For linearly elastic (Hookean) behavior, it follows that the principal stresses can be computed from the principal strains (shear strain is zero for this axis system):

$$\sigma_1 = \frac{E}{(1-\nu^2)} (\varepsilon_1 + \nu \varepsilon_2)$$

$$\sigma_2 = \frac{E}{(1-\nu^2)} (\varepsilon_2 + \nu \varepsilon_1)$$
(6)

And it should also be noted that for Hookean materials the principal strain and stress directions coincide, so the results for the angle, ϕ , are unchanged for principal stress directions.