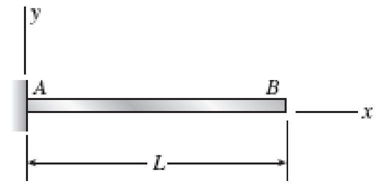
Moment-Curvature Relationships and Differential Equations for a Beam

Problem 9.2-3 The deflection curve for a cantilever beam AB (see figure) is given by the following equation:

$$v = -\frac{q_0 x^2}{120LEI}(10L^3 - 10L^2x + 5Lx^2 - x^3)$$

Describe the load acting on the beam.



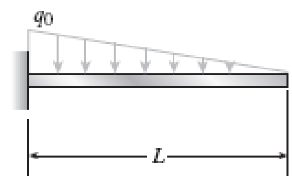
Probs. 9.2-3 and 9.2-4

Solution 9.2-3 **Cantilever beam**

$$v = -\frac{q_0 x^2}{120\,LEI}(10\,L^3 - 10\,L^2 x + 5\,Lx^2 - x^3)$$

Take four consecutive derivatives and obtain:

$$v'''' = -\frac{q_0}{LEI}(L - x)$$



From Eq. (9-12c):

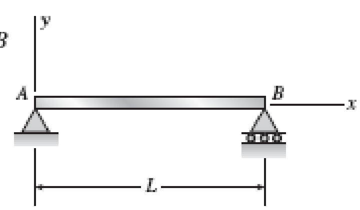
$$q = -EIv'''' = q_0\left(1 - \frac{x}{L}\right) \quad \leftarrow$$

The load is a downward triangular load of maximum intensity q_0. ←

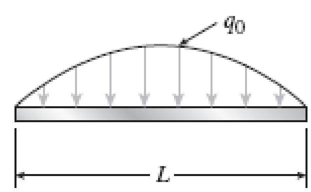
Problem 9.2-2 The deflection curve for a simple beam AB is given by the following equation:

$$v = -\frac{q_0 L^4}{\pi^4 EI}\sin\frac{\pi x}{L}$$

(a) Describe the load acting on the beam.
(b) Determine the reactions R_A and R_B at the supports.
(c) Determine the maximum bending moment M_{max}.



Probs. 9.2-1 and 9.2-2



$$v = -\frac{q_0 L^4}{\pi^4 EI}\sin\frac{\pi x}{L}$$

$$v' = -\frac{q_0 L^3}{\pi^3 EI}\cos\frac{\pi x}{L}$$

$$v'' = \frac{q_0 L^2}{\pi^2 EI}\sin\frac{\pi x}{L}$$

$$v''' = \frac{q_0 L}{\pi EI}\cos\frac{\pi x}{L}$$

$$v'''' = -\frac{q_0}{EI}\sin\frac{\pi x}{L}$$

(a) Load (Eq. 9-12c)

$$q = -EIv'''' = q_0 \sin\frac{\pi x}{L} \quad \leftarrow$$

(b) Reactions (Eq. 9-12b)

$$V = EIv''' = \frac{q_0 L}{\pi}\cos\frac{\pi x}{L}$$

At $x = 0$: $V = R_A = \frac{q_0 L}{\pi}$ ←

At $x = L$: $V = -R_B = -\frac{q_0 L}{\pi}$; $R_B = \frac{q_0 L}{\pi}$ ←

(c) Maximum bending moment (Eq. 9-12a)

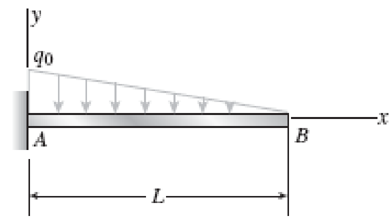
$$M = EIv'' = \frac{q_0 L^2}{\pi^2}\sin\frac{\pi x}{L}$$

For maximum moment, $x = \frac{L}{2}$; $M_{max} = \frac{q_0 L^2}{\pi^2}$ ←

Deflection by Successive Integrations

Problem 9.3-10 A cantilever beam AB supporting a triangularly distributed load of maximum intensity q_0 is shown in the figure.

Derive the equation of the deflection curve and then obtain formulas for the deflection δ_B and angle of rotation θ_B at the free end. (*Note:* Use the second-order differential equation of the deflection curve.)



Solution 9.3-10 **Cantilever beam (triangular load)**

BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = -\frac{q_0}{6L}(L - x)^3$$

$$EIv' = \frac{q_0}{24L}(L - x)^4 + C_1$$

B.C. $v'(0) = 0 \quad \therefore C_1 = -\frac{q_0 L^3}{24}$

$$EIv = -\frac{q_0}{120L}(L - x)^5 - \frac{q_0 L^3 x}{24} + C_2$$

B.C. $v(0) = 0 \quad \therefore C_2 = \frac{q_0 L^4}{120}$

$$v = -\frac{q_0 x^2}{120\,LEI}(10L^3 - 10L^2 x + 5Lx^2 - x^3) \quad \longleftarrow$$

$$v' = -\frac{q_0 x}{24\,LEI}(4L^3 - 6L^2 x + 4Lx^2 - x^3)$$

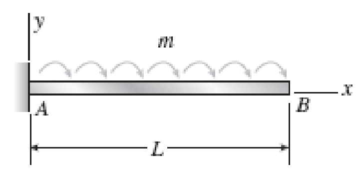
$$\delta_B = -v(L) = \frac{q_0 L^4}{30\,EI} \quad \longleftarrow$$

$$\theta_B = -v'(L) = \frac{q_0 L^3}{24\,EI} \quad \longleftarrow$$

(These results agree with Case 8, Table G-1.)

Problem 9.3-11 A cantilever beam AB is acted upon by a uniformly distributed moment (bending moment, not torque) of intensity m per unit distance along the axis of the beam (see figure).

Derive the equation of the deflection curve and then obtain formulas for the deflection δ_B and angle of rotation θ_B at the free end. (*Note:* Use the second-order differential equation of the deflection curve.)



Solution 9.3-11 **Cantilever beam (distributed moment)**

BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = -m(L - x)$$

$$EIv' = -m\left(Lx - \frac{x^2}{2}\right) + C_1$$

B.C. $v'(0) = 0 \quad \therefore C_1 = 0$

$$EIv = -m\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) + C_2$$

B.C. $v(0) = 0 \quad \therefore C_2 = 0$

$$v = -\frac{mx^2}{6\,EI}(3L - x) \quad \longleftarrow$$

$$v' = -\frac{mx}{2EI}(2L - x)$$

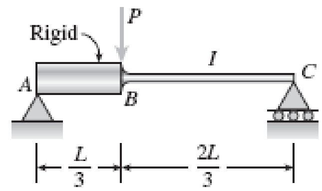
$$\delta_B = -v(L) = \frac{mL^3}{3\,EI} \quad \longleftarrow$$

$$\theta_B = -v'(L) = \frac{mL^2}{2\,EI} \quad \longleftarrow$$

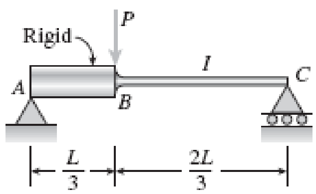
Non-Prismatic Members

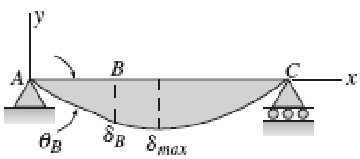
Problem 9.7-4 A beam ABC has a rigid segment from A to B and a flexible segment with moment of inertia I from B to C (see figure). A concentrated load P acts at point B.

Determine the angle of rotation θ_A of the rigid segment, the deflection δ_B at point B, and the maximum deflection δ_{max}.



Solution 9.7-4 **Simple beam with a rigid segment**





From A to B

$$v = -\frac{3\delta_B x}{L} \quad \left(0 \le x \le \frac{L}{3}\right) \qquad (1)$$

$$v' = -\frac{3\delta_B}{L} \quad \left(0 \le x \le \frac{L}{3}\right) \qquad (2)$$

From B to C

$$EIv'' = M = \frac{PL}{3} - \frac{Px}{3} \qquad (3)$$

$$EIv' = \frac{PLx}{3} - \frac{Px^2}{6} + C_1$$

B.C. 1 At $x = L/3$, $v' = -\frac{3\delta_B}{L}$

$$\therefore\ C_1 = -\frac{5PL^2}{54} - \frac{3EI\delta_B}{L}$$

$$EIv' = \frac{PLx}{3} - \frac{Px^2}{6} - \frac{5PL^2}{54} - \frac{3EI\delta_B}{L} \quad \left(\frac{L}{3} \le x \le L\right) \qquad (4)$$

$$EIv = \frac{PLx^2}{6} - \frac{Px^3}{18} - \frac{5PL^2x}{54} - \frac{3EI\delta_B x}{L} + C_2 \quad \left(\frac{L}{3} \le x \le L\right)$$

B.C. 2 $v(L) = 0$ $\quad \therefore\ C_2 = -\frac{PL^3}{54} + 3EI\delta_B$

$$EIv = \frac{PLx^2}{6} - \frac{Px^3}{18} - \frac{5PL^2x}{54} - \frac{3EI\delta_B x}{L} - \frac{PL^2}{54} + 3EI\delta_B \quad \left(\frac{L}{3} \le x \le L\right) \qquad (5)$$

B.C. 3 At $x = \frac{L}{3}$, $(v_B)_{\text{Left}} = (v_B)_{\text{Right}}$ (Eqs. 1 and 5)

$$\therefore\ \delta_B = \frac{8PL^3}{729EI} \quad \longleftarrow$$

$$\theta_A = \frac{\delta_B}{L/3} = \frac{8PL^2}{243EI} \quad \longleftarrow$$

Substitute for δ_B in Eq. (5) and simplify:

$$v = \frac{P}{486EI}(7L^3 - 61L^2x + 81Lx^2 - 27x^3) \quad \left(\frac{L}{3} \le x \le L\right) \qquad (6)$$

Also,

$$v' = \frac{P}{486EI}(-61L^2 + 162Lx - 81x^2) \quad \left(\frac{L}{3} \le x \le L\right) \qquad (7)$$

Maximum deflection

$v' = 0$ gives $x_1 = \frac{L}{9}(9 - 2\sqrt{5}) = 0.5031L$

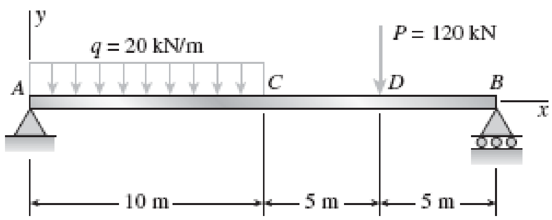
Substitute x_1 in Eq. (6) and simplify:

$$v_{max} = -\frac{40\sqrt{5}PL^3}{6561EI}$$

$$\delta_{max} = -v_{max} = \frac{40\sqrt{5}PL^3}{6561EI} = 0.01363\frac{PL^3}{EI} \quad \longleftarrow$$

Singularity Functions

Solution 9.12-10 Simple beam



$q = 20$ kN/m
$P = 120$ kN
$\frac{L}{2} = 10$ m
$L = 20$ m
$E = 200$ GPa
$I = 2.60 \times 10^{-3}$ m^4

FROM PROB. 9.11-10: Units: kilonewtons, meters

$$EIv'''' = -q(x) = 180\langle x\rangle^{-1} - 20\langle x\rangle^0 + 20\langle x-10\rangle^0 - 120\langle x-15\rangle^{-1} + 140\langle x-20\rangle^{-1}$$

Note: $\langle x-20\rangle^{-1} = 0$ and may be dropped from the equation

INTEGRATE THE EQUATION

$$EIv''' = V = 180\langle x\rangle^0 - 20\langle x\rangle^1 + 20\langle x-10\rangle^1 - 120\langle x-15\rangle^0$$

Note: $\langle x\rangle^0 = 1$ and $\langle x\rangle^1 = x$

$$EIv'' = M = 180x - 20(x^2/2) + (20/2)\langle x-10\rangle^2 - 120\langle x-15\rangle^1$$

$$EIv' = 180(x^2/2) - 20(x^3/6) + (10/3)\langle x-10\rangle^3 - 60\langle x-15\rangle^2 + C_1$$

$$EIv = 30x^3 - (5/6)x^4 + (5/6)\langle x-10\rangle^4 - 20\langle x-15\rangle^3 + C_1x + C_2$$

B.C. $EIv(0) = 0 \quad 0 = 0 - 0 + 0 - 0 + C_1(0) + C_2$

$\therefore C_2 = 0$

B.C. $EIv(20) = 0$

$$0 = 30(20)^3 - (5/6)(20)^4 + (5/6)(10)^4 - 20(5)^3 + C_1(20)$$

$$0 = 112{,}500 + 20C_1 \quad \therefore C_1 = -5625$$

FINAL EQUATIONS

$$EIv' = 90x^2 - (10/3)x^3 + (10/3)\langle x-10\rangle^3 - 60\langle x-15\rangle^2 - 5625$$

$$EIv = 30x^3 - (5/6)x^4 + (5/6)\langle x-10\rangle^4 - 20\langle x-15\rangle^3 - 5625x \quad \longleftarrow$$

(x = meters, v = meters, v' = radians, E = kilopascals, I = meters^4)

θ_B = COUNTERCLOCKWISE ROTATION AT SUPPORT B ($x = 20$)

$$EIv'(20) = 90(20)^2 - (10/3)(20)^3 + (10/3)(10)^3 - 60(5)^2 - 5625 = 5541.67$$

$$\theta_B = v'(20) = \frac{5541.67}{EI} = \frac{5541.67}{(200 \times 10^6 \text{ kPa})(2.60 \times 10^{-3} \text{ m})}$$

$$= 0.01066 \text{ rad} \quad \text{(counterclockwise)} \quad \longleftarrow$$

δ_D = DOWNWARD DEFLECTION AT POINT D ($x = 15$)

$$EIv(15) = 30(15)^3 - (5/6)(15)^4 + (5/6)(5)^4 - 20(0) - 5625(15) = -24{,}791.7$$

$$\delta_D = -v(15) = \frac{24{,}791.7}{EI} = \frac{24{,}791.7}{(200 \times 10^6 \text{ kPa})(2.60 \times 10^{-3} \text{ m})}$$

$$= 0.04768 \text{ m} = 47.68 \text{ mm} \quad \text{(downward)} \quad \longleftarrow$$

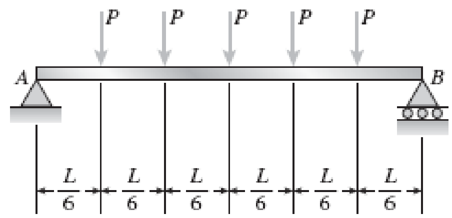
Method of Superposition

Problem 9.5-2 A simple beam AB supports five equally spaced loads P (see figure).

(a) Determine the deflection δ_1 at the midpoint of the beam.

(b) If the same total load ($5P$) is distributed as a uniform load on the beam, what is the deflection δ_2 at the midpoint?

(c) Calculate the ratio of δ_1 to δ_2.



Solution 9.5-2 **Simple beam with 5 loads**

(a) Table G-2, Cases 4 and 6

$$\delta_1 = \frac{P\left(\frac{L}{6}\right)}{24EI}\left[3L^2 - 4\left(\frac{L}{6}\right)^2\right] + \frac{P\left(\frac{L}{3}\right)}{24EI}\left[3L^2 - 4\left(\frac{L}{3}\right)^2\right] + \frac{PL^3}{48EI} = \frac{11PL^3}{144EI} \longleftarrow$$

(b) Table G-2, Case 1 $qL = 5P$

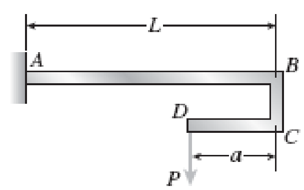
$$\delta_2 = \frac{5qL^4}{384EI} = \frac{25PL^3}{384EI} \longleftarrow$$

(c) $\frac{\delta_1}{\delta_2} = \frac{11}{144}\left(\frac{384}{25}\right) = \frac{88}{75} = 1.173 \longleftarrow$

Problem 9.5-3 The cantilever beam AB shown in the figure has an extension BCD attached to its free end. A force P acts at the end of the extension.

(a) Find the ratio a/L so that the vertical deflection of point B will be zero.

(b) Find the ratio a/L so that the angle of rotation at point B will be zero.



Solution 9.5-3 **Cantilever beam with extension**

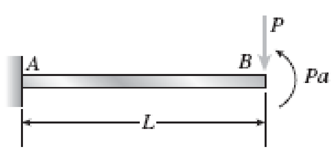


Table G-1, Cases 4 and 6

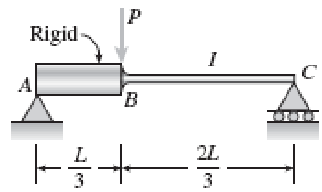
(a) $\delta_B = \frac{PL^3}{3EI} - \frac{PaL^2}{2EI} = 0 \quad \frac{a}{L} = \frac{2}{3} \longleftarrow$

(b) $\theta_B = \frac{PL^2}{2EI} - \frac{PaL}{EI} = 0 \quad \frac{a}{L} = \frac{1}{2} \longleftarrow$

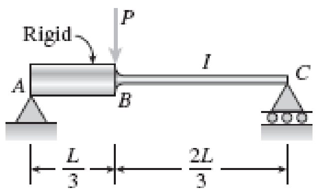
Non-Prismatic Members

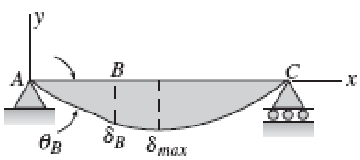
Problem 9.7-4 A beam ABC has a rigid segment from A to B and a flexible segment with moment of inertia I from B to C (see figure). A concentrated load P acts at point B.

Determine the angle of rotation θ_A of the rigid segment, the deflection δ_B at point B, and the maximum deflection δ_{max}.



Solution 9.7-4 **Simple beam with a rigid segment**





FROM A TO B

$$v = -\frac{3\delta_B x}{L} \quad \left(0 \leq x \leq \frac{L}{3}\right) \tag{1}$$

$$v' = -\frac{3\delta_B}{L} \quad \left(0 \leq x \leq \frac{L}{3}\right) \tag{2}$$

FROM B TO C

$$EIv'' = M = \frac{PL}{3} - \frac{Px}{3} \tag{3}$$

$$EIv' = \frac{PLx}{3} - \frac{Px^2}{6} + C_1$$

B.C. 1 At $x = L/3$, $v' = -\frac{3\delta_B}{L}$

$$\therefore C_1 = -\frac{5PL^2}{54} - \frac{3EI\delta_B}{L}$$

$$EIv' = \frac{PLx}{3} - \frac{Px^2}{6} - \frac{5PL^2}{54} - \frac{3EI\delta_B}{L} \quad \left(\frac{L}{3} \leq x \leq L\right) \tag{4}$$

$$EIv = \frac{PLx^2}{6} - \frac{Px^3}{18} - \frac{5PL^2x}{54} - \frac{3EI\delta_B x}{L} + C_2 \quad \left(\frac{L}{3} \leq x \leq L\right)$$

B.C. 2 $v(L) = 0$ $\therefore C_2 = -\frac{PL^3}{54} + 3EI\delta_B$

$$EIv = \frac{PLx^2}{6} - \frac{Px^3}{18} - \frac{5PL^2x}{54} - \frac{3EI\delta_B x}{L} - \frac{PL^3}{54} + 3EI\delta_B \quad \left(\frac{L}{3} \leq x \leq L\right) \tag{5}$$

B.C. 3 At $x = \frac{L}{3}$, $(v_B)_{\text{Left}} = (v_B)_{\text{Right}}$ (Eqs. 1 and 5)

$$\therefore \delta_B = \frac{8PL^3}{729EI} \quad \longleftarrow$$

$$\theta_A = \frac{\delta_B}{L/3} = \frac{8PL^2}{243EI} \quad \longleftarrow$$

Substitute for δ_B in Eq. (5) and simplify:

$$v = \frac{P}{486EI}(7L^3 - 61L^2x + 81Lx^2 - 27x^3) \quad \left(\frac{L}{3} \leq x \leq L\right) \tag{6}$$

Also,

$$v' = \frac{P}{486EI}(-61L^2 + 162Lx - 81x^2) \quad \left(\frac{L}{3} \leq x \leq L\right) \tag{7}$$

MAXIMUM DEFLECTION

$v' = 0$ gives $x_1 = \frac{L}{9}(9 - 2\sqrt{5}) = 0.5031L$

Substitute x_1 in Eq. (6) and simplify:

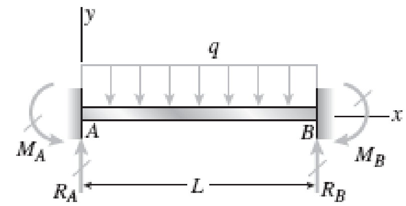
$$v_{max} = -\frac{40\sqrt{5}PL^3}{6561EI}$$

$$\delta_{max} = -v_{max} = \frac{40\sqrt{5}PL^3}{6561EI} = 0.01363\frac{PL^3}{EI} \quad \longleftarrow$$

Statically Indeterminate Beams

Problem 10.3-2 A fixed-end beam *AB* of length *L* supports a uniform load of intensity *q* (see figure).

Beginning with the second-order differential equation of the deflection curve (the bending-moment equation), obtain the reactions, shear forces, bending moments, slopes, and deflections of the beam. Construct the shear-force and bending-moment diagrams, labeling all critical ordinates.



Solution 10.3-2 Fixed-end beam (uniform load)

Select M_A as the redundant reaction.

REACTIONS (FROM SYMMETRY AND EQUILIBRIUM)

$$R_A = R_B = \frac{qL}{2} \qquad M_B = M_A$$

BENDING MOMENT (FROM EQUILIBRIUM)

$$M = R_A x - M_A - \frac{qx^2}{2} = -M_A + \frac{q}{2}(Lx - x^2) \qquad (1)$$

DIFFERENTIAL EQUATIONS

$$EIv'' = M = -M_A + \frac{q}{2}(Lx - x^2)$$

$$EIv' = -M_A x + \frac{q}{2}\left(\frac{Lx^2}{2} - \frac{x^3}{3}\right) + C_1 \qquad (2)$$

B.C. 1 $v'(0) = 0 \quad \therefore C_1 = 0$

$$EIv = -\frac{M_A x^2}{2} + \frac{q}{2}\left(\frac{Lx^3}{6} - \frac{x^4}{12}\right) + C_2 \qquad (3)$$

B.C. 2 $v(0) = 0 \quad \therefore C_2 = 0$

B.C. 3 $v(L) = 0 \quad \therefore M_A = \frac{qL^2}{12}$

REACTIONS

$$R_A = R_B = \frac{qL}{2} \quad M_A = M_B = \frac{qL^2}{12} \quad \longleftarrow$$

SHEAR FORCE (FROM EQUILIBRIUM)

$$V = R_A - qx = \frac{q}{2}(L - 2x) \quad \longleftarrow$$

BENDING MOMENT (FROM EQ. 1)

$$M = -\frac{q}{12}(L^2 - 6Lx + 6x^2) \quad \longleftarrow$$

SLOPE (FROM EQ. 2)

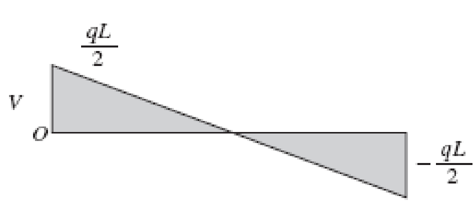
$$v' = -\frac{qx}{12EI}(L^2 - 3Lx + 2x^2) \quad \longleftarrow$$

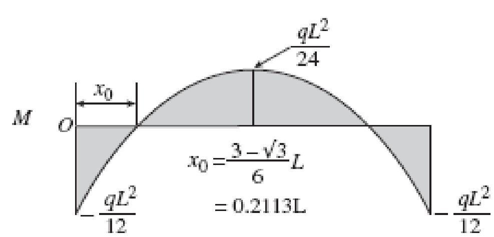
DEFLECTION (FROM EQ. 3)

$$v = -\frac{qx^2}{24EI}(L - x)^2 \quad \longleftarrow$$

$$\delta_{max} = -v\left(\frac{L}{2}\right) = \frac{qL^4}{384EI}$$

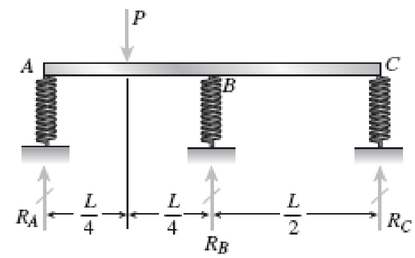
SHEAR-FORCE AND BENDING-MOMENT DIAGRAMS



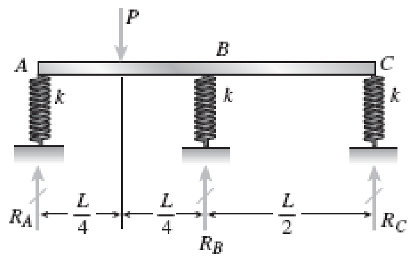


Problem 10.4-21 A wide-flange beam ABC rests on three identical spring supports at points A, B, and C (see figure). The flexural rigidity of the beam is $EI = 6912 \times 10^6$ lb-in.^{2}, and each spring has stiffness $k = 62{,}500$ lb/in. The length of the beam is $L = 16$ ft.

If the load P is 6000 lb, what are the reactions R_A, R_B, and R_C? Also, draw the shear-force and bending-moment diagrams for the beam, labeling all critical ordinates.



Solution 10.4-21 **Beam on three springs**

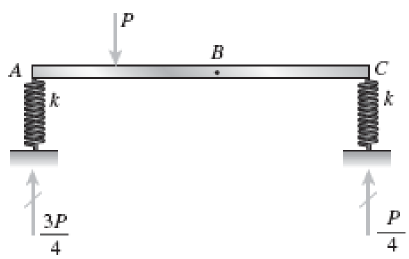


Select R_B as redundant.

EQUILIBRIUM

$$R_A = \frac{3P}{4} - \frac{R_B}{2} \qquad R_C = \frac{P}{4} - \frac{R_B}{2}$$

RELEASED STRUCTURE AND FORCE-DISPL. EQS.



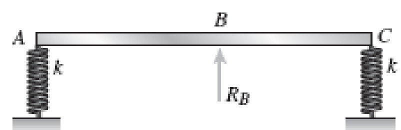
$$(\delta_A)_1 = \frac{3P}{4k}$$

$$(\delta_C)_1 = \frac{P}{4k}$$

$$(\delta_B)_1 = \frac{1}{2}[(\delta_A)_1 + (\delta_C)_1] + \frac{P\left(\frac{L}{4}\right)\left[3L^2 - 4\left(\frac{L}{4}\right)^2\right]}{48EI}$$

(Case 5, Table G-2)

$$(\delta_B)_1 = \frac{P}{2k} + \frac{11PL^3}{768EI} \quad \text{(downward)}$$



$$(\delta_A)_2 = \frac{R_B}{2k}$$

$$(\delta_C)_2 = \frac{R_B}{2k}$$

$$(\delta_B)_2 = \frac{1}{2}[(\delta_A)_2 + (\delta_C)_2] + \frac{R_B L^3}{48EI}$$

$$= \frac{R_B}{2k} + \frac{R_B L^3}{48EI} \quad \text{(upward)}$$

COMPATIBILITY $\quad (\delta_B)_1 - (\delta_B)_2 = \frac{R_B}{k}$

Substitute and solve:

$$R_B = P\left(\frac{384EI + 11kL^3}{1152EI + 16kL^3}\right)$$

Let $k^* = \frac{kL^3}{EI}$ (nondimensional) ←

$$R_B = \frac{P}{16}\left(\frac{384 + 11k^*}{72 + k^*}\right) \quad \leftarrow$$

FROM EQUILIBRIUM:

$$R_A = \frac{P}{32}\left(\frac{1344 + 13k^*}{72 + k^*}\right) \quad \leftarrow$$

$$R_C = \frac{3P}{32}\left(\frac{64 - k^*}{72 + k^*}\right) \quad \leftarrow$$

NUMERICAL VALUES

$EI = 6912 \times 10^6$ lb-in.^{2} $\quad k = 62{,}500$ lb/in.

$L = 16$ ft $= 192$ in. $\quad P = 6000$ lb

$k^* = \frac{kL^3}{EI} = 64 \quad R_B = 3000$ lb ←

$R_A = 3000$ lb $\quad R_C = 0$ ←

SHEAR-FORCE AND BENDING-MOMENT DIAGRAMS