Real engineers actually laugh at jokes about mathematicians.

- The Euler formula that was developed earlier was based on the assumption that the concentrated compressive load *P* on the column acts though the centroid of the cross section of the column (Fig. 1).
- In many realistic situations, however, this is not the case. The load *P* applied to a column is never perfectly centric.

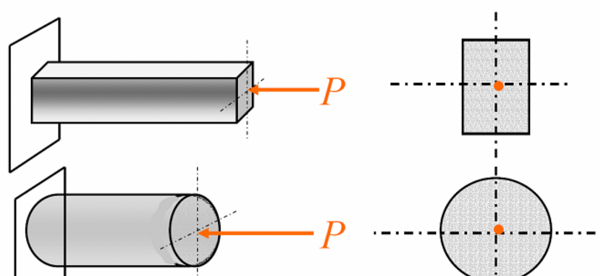


Figure 1. Centric Loading

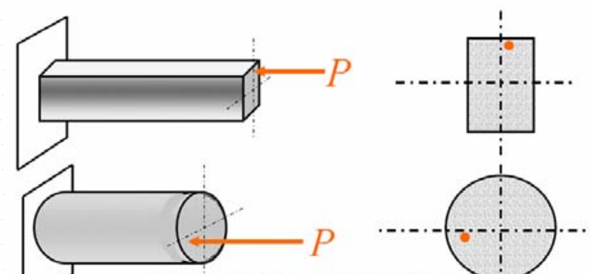
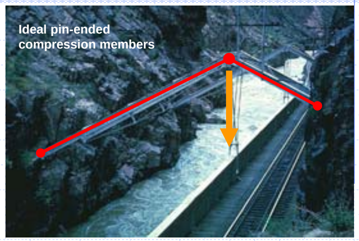


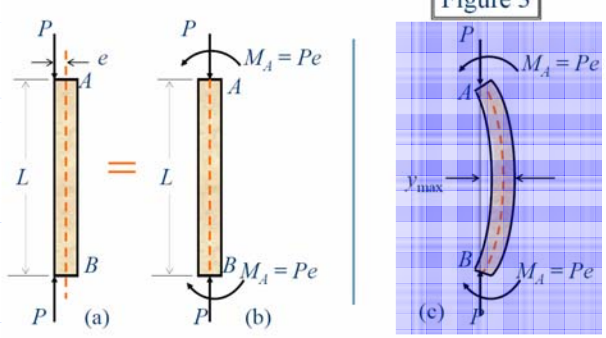
Figure. 2. Eccentric Loading



- In many structures and buildings, compressive members that used as columns are subjected to eccentric concentrated compressive loads (Fig. 2) as well as moments.
- The difference between a column and a beam is that in a column, the magnitude of P is much much greater than that of a beam.

- Denoting by e the eccentricity of the load, that is, the distance between the line of action of P and the axis of the column, as shown in Fig. 3a, the given eccentric load can be replaced by a centric force P and a couple $M_A = Pe$ (Fig. 3b).
- It is clear that no matter how small P and e, the couple M_A will cause bending in the column, as shown in Fig. 3c.

Figure 3



- As the eccentric load is increased, both the couple M_A and the axial force P increase, and both cause the column to bend further.
- Viewed in this way, the problem of buckling is not a question of determining how long the column can remain straight and stable under an increasing load, but rather how much it can be permitted to bend under the

– The increasing load, if the allowable stress is not to be exceeded and if the deflection y_{max} is not to become excessive.

– The deflection equation (elastic curve) for this column can written and solved in a manner similar to column subjected to centric loading *P*.

– Derivation of the formula:

- Drawing the free-body diagram of a portion *AQ* of the column of Figure 3 and choosing the coordinate axes as shown in Fig. 4, the bending moment at *Q* is given by

$$M(x) = -Py - M_A = -Py - Pe \qquad (1)$$

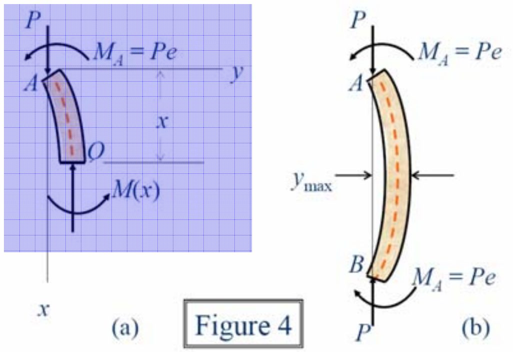


Figure 4

– Derivation of the formula (cont'd):

- Recalling that the relationship between the curvature and the moment along the column is given by

$$EI\frac{d^2y}{dx^2} = M(x) \qquad (2)$$

- Therefore, combining Eqs. 1 and 2, yields

$$\frac{d^2y}{dx^2} = -\frac{P}{EI}y - \frac{Pe}{EI} \qquad (3)$$

– Derivation of the formula (cont'd):

- Moving the term containing *y* in Eq. 3 to the left and setting

$$p^2 = \frac{P}{EI} \qquad (4)$$

as done earlier, Eq. 3 can be written as

$$\frac{d^2y}{dx^2} + p^2 y = -p^2 e \qquad (5)$$

– Derivation of the formula (cont'd):

- The general solution of the differential equation (Eq. 5) is

$$y = A \sin px + B \cos px - e \qquad (6)$$

- Using the boundary condition *y* = 0, at *x* = 0, Eq. 6 gives

$$B = e \qquad (7)$$

– Derivation of the formula (cont'd):

- Using the other boundary condition at the other end: *y* = 0, at *x* = *L*, Eq. 6 gives

$$A \sin pL + = e(1 - \cos pL) \qquad (8)$$

- Recalling that

$$\sin pL = 2 \sin \frac{pL}{2} \cos \frac{pL}{2} \qquad (9)$$

and

$$1 - \cos pL = 2 \sin^2 \frac{pL}{2} \qquad (10)$$

– Derivation of the formula (cont'd):

- Substituting Eqs. 9 and 10 into Eq. 8, we obtain

$$A = e \tan \frac{pL}{2} \qquad (11)$$

- And substituting for *A* and *B* into Eq. 6, the elastic curve can be obtained as

$$y = e\left[\tan \frac{pL}{2} \sin px + \cos px - 1\right] \qquad (12)$$

(with $\sec(x) = \frac{1}{\cos(x)}$)

– Derivation of the formula (cont'd):

- The maximum deflection is obtained by setting $x = L/2$ in Eq. 12. The result is

$$y_{max} = e\left[\sec\sqrt{\frac{P}{EI}}\frac{L}{2} - 1\right] \qquad (13)$$

- y_{max} becomes infinite when the argument of the secant function in Eq. 13 equals $\pi/2$. While the deflection does not become infinite, it nevertheless becomes unacceptably large, and P would reach the critical value P_{cr}.

– Derivation of the formula (cont'd):

- Based on that, when the argument of the secant function equals $\pi/2$, that is

$$\sqrt{\frac{P}{EI}}\frac{L}{2} = \frac{\pi}{2} \qquad (14)$$

- Therefore, the critical load is obtained as

$$P_{cr} = \frac{\pi^2 EI}{L^2} \qquad (15)$$

which is the same for column under centric loading.

– Derivation of the formula (cont'd):

- Solving Eq. 15 for EI and substituting into Eq. 13, the maximum deflection can be expressed in an alternative form as

$$y_{max} = e\left[\sec\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}} - 1\right] \qquad (16)$$

- The maximum stress occur at midspan of the column (at $x = L/2$), and can computed from

$$\sigma_{max} = \frac{P}{A} + \frac{M_{max}c}{I} \qquad (17)$$

– Derivation of the formula (cont'd):

- The moment at the midspan of the column is maximum (M_{max}) and is given by

$$M_{max} = Py_{max} + M_A = P(y_{max} + e) \qquad (18)$$

- Substituting this value of M_{max} and the expression for y_{max} of Eq. 13 into Eq. 17 and noting that $I = Ar^2$, the result is

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right)\right] \qquad (19)$$

– Derivation of the formula (cont'd):

- An alternative form of Eq. 19 is obtained by substituting for y_{max} from Eq. 16 into Eq. 18. Thus

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right] \qquad (20)$$

- Where P_{cr} is Euler buckling load for centric loading as given by Eq. 15.

- Notes on Equations 19 and 20:
  1. Since σ_{max} does not vary linearly with P, the principle of superposition does not apply to the determination of the stress due to the simultaneous application of several loads.
  2. The resultant load must first be computed and then Eqs. 19 or 20 may be used to determine the corresponding stress.
  3. Consequently, any given factor of safety should be applied to the load and not to the stress.

– Derivation of the formula (cont'd):

- If we substitute for $I = Ar^2$ in Eq. 19, and solve for P/A in front of the bracket, the *secant formula* is obtained:

$$\frac{P}{A} = \frac{\sigma_{max}}{1 + \frac{ec}{r^2}\sec\left(\frac{1}{2}\sqrt{\frac{P}{EA}}\frac{L'}{r}\right)} \qquad (21)$$

Where L' is used to make the formula applicable to various end conditions.

The secant formula for a column subjected to eccentric compressive load P is given by

$$\frac{P}{A} = \frac{\sigma_{max}}{1 + \frac{ec}{r^2}\sec\left(\frac{1}{2}\sqrt{\frac{P}{EA}}\frac{L'}{r}\right)} \qquad (21)$$

e = eccentricity A = area of cross section
E = modulus of elasticity r = min radius of gyration
c = distance from $N.A.$ to extreme fiber.
L' = effective length for column

An alternative form for the secant formula is give by

$$P = \frac{\sigma_{max} A}{1 + \frac{ec}{r^2} \sec\left(\frac{1}{2}\sqrt{\frac{P}{EA}}\frac{L'}{r}\right)} \qquad (22)$$

e = eccentricity A = area of cross section
E = modulus of elasticity r = min radius of gyration
c = distance from $N.A.$ to extreme fiber.
L'= effective length for column

– General Notes on the Secant Formula:
- The formula of Eq. 21 is referred to as the secant formula; it defines the force per unit area, P/A, which causes a specified maximum stress σ_{max} in a column of given effective slenderness ratio, L'/r, for a given value of the ratio ec/r^2, where e is the eccentricity of the applied load.
- Since P or P/A appears in both sides of Eqs. 21 or 22, it necessary to solve these equations by

by an iterative procedure (trial and error) to obtain the value of P or P/A corresponding to a given column and loading condition.
- The curves shown in Figs. 5 and 6 are constructed for steel column using Eq. 21.
- These curves make it possible to determine the load per unit area P/A, which causes the column to yield for given values of the ratio L'/r and ec/r^2.

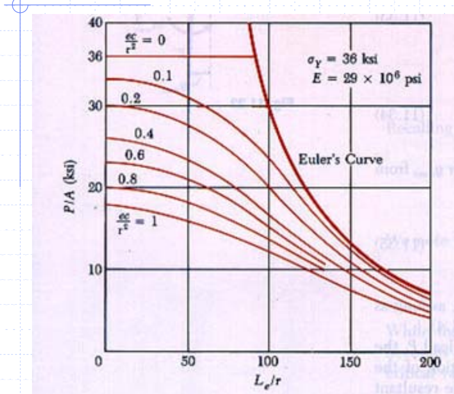


Figure 5.
Load per unit area, P/A, causing yield in column (US Customary units)
(Beer & Johnston 1992)

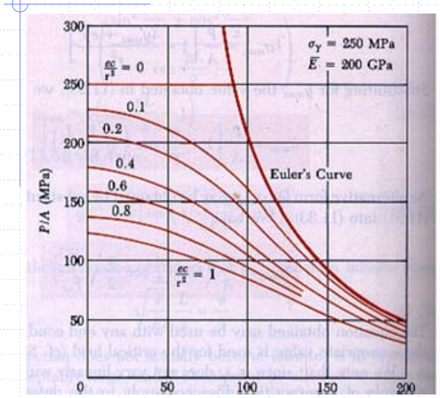


Figure 6.
Load per unit area, P/A, causing yield in column (SI)

(Beer & Johnston 1992)

Remember: 2 plus 2 is four, but adding a safety factor we will call it 5.

It should be noted that for small values of L'/r, the secant is almost equal to unity in Eqs. 21 and 22, and *P/A* (or *P*) may be assumed equal to

$$\frac{P}{A} = \frac{\sigma_{max}}{1+\frac{ec}{r^2}} \quad \text{or} \quad P = \frac{\sigma_{max} A}{1+\frac{ec}{r^2}} \qquad (23)$$

– Notes on Figures 5 and 6:
- For large value of L'/r, the curves corresponding to the various values of the ratio ec/r^2 get very close to Euler's curve, and thus that the effect of the eccentricity of the loading on the value of *P/A* becomes negligible.
- The secant formula is mainly useful for intermediate values of L'/r.

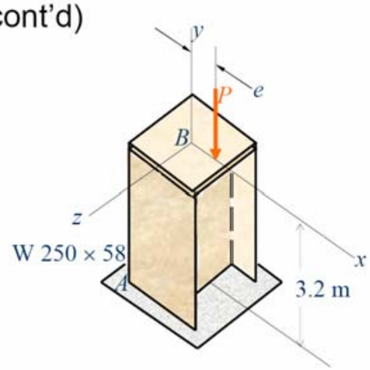
Did they know that?

- Example 1

  The axial load P is applied at a point located on the x axis at a distance e from the geometric axis of the W 250 × 58 rolled-steel column AB (Fig. 7). When P = 350 kN, it is observed that the horizontal deflection of the top of the column is 5 mm. Using E = 200 GPa, determine (a) the eccentricity e of the load, (b) the maximum stress in the column.

- Example 1 (cont'd)

Figure 7



- For W 250 × 58

  $I_x = 87.0\times10^{-6}\ \mathrm{m}^4 \qquad A = 7.42\times10^{-3}\ \mathrm{m}^2$

  $I_y = 18.73\times10^{-6}\ \mathrm{m}^4 \qquad r_x = 0.1085\ \mathrm{m}$

  $S_y = 184.5\times10^{-6}\ \mathrm{m}^3 \qquad r_y = 0.0502\ \mathrm{m}$

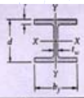
- One-end fixed, one-end free column,

$$L' = 2(3.2) = 6.4\ \mathrm{m}$$

- Therefore

$$P_{cr} = \frac{\pi^2 EI_y}{L'^2} = \frac{\pi(200\times10^9)(18.73\times10^{-6})}{(6.4)^2} = 902.6\ \mathrm{kN}$$

Appendix C. Properties of Rolled-Steel Shapes
(SI Units)
Continued from page 705



W Shapes
(Wide-Flange Shapes)

Designation†	Area A, mm²	Depth d, mm	Flange Width b_f, mm	Flange Thickness t_f, mm	Web Thickness t_w, mm	Axis X-X I_x 10^6 mm⁴	Axis X-X S_x 10^3 mm³	Axis X-X r_x mm	Axis Y-Y I_y 10^6 mm⁴	Axis Y-Y S_y 10^3 mm³	Axis Y-Y r_y mm
W310 × 143	18200	323	309	22.9	14.0	347	2150	138.2	112.4	728	78.5
107	13600	311	306	17.0	10.9	248	1595	134.9	81.2	531	77.2
74	9480	310	205	16.3	9.4	164.0	1058	131.6	23.4	228	49.8
60	7610	303	203	13.1	7.5	129.0	851	130.3	18.36	180.9	49.0
52	6650	317	167	13.2	7.6	118.6	748	133.4	10.20	122.2	39.1
44.5	5670	313	166	11.2	6.6	99.1	633	132.3	8.45	101.8	38.6
38.7	4940	310	165	9.7	5.8	84.9	545	131.3	7.20	87.3	38.4
32.7	4180	313	102	10.8	6.6	64.9	415	124.7	1.940	38.0	21.5
23.8	3040	305	101	6.7	5.6	42.9	281	118.6	1.174	23.2	19.63
W250 × 167	21200	289	265	31.8	19.2	298.0	2060	118.4	98.2	741	68.1
101	12900	264	257	19.6	11.9	164.0	1242	112.8	55.8	434	65.8
80	10200	256	255	15.6	9.4	126.1	985	111.0	42.8	336	65.0
67	8580	257	204	15.7	8.9	103.2	803	110.0	22.2	218	51.1
58	7420	252	203	13.5	8.0	87.0	690	108.5	18.73	184.5	50.3
49.1	6260	247	202	11.0	7.4	70.8	573	106.4	15.23	150.8	49.3
44.8	5700	266	148	13.0	7.6	70.8	532	111.3	6.95	93.9	34.8
32.7	4190	258	146	9.1	6.1	49.1	381	108.5	4.75	65.1	33.8
28.4	3630	260	102	10.0	6.4	40.1	308	105.2	1.796	35.2	22.2
22.3	2850	254	102	6.9	5.8	28.7	226	100.3	1.203	23.6	20.6

- (a) From Eq. 16,

$$y_{\max} = e\left[\sec\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}-1\right]$$

$$0.005 = e\left[\sec\frac{\pi}{2}\sqrt{\frac{350}{902.6}}-1\right] = e[\sec(0.9782)-1]$$

$$0.005 = e\left[\frac{1}{\cos(0.9782)}-1\right] \Rightarrow \quad e = 0.00633\ \text{m} = \boxed{6.33\ \text{mm}}$$

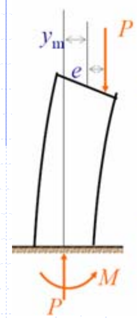
- (b) From Eq. 20,

$$\sigma_{\max} = \frac{P}{A}\left[1+\frac{ec}{r^2}\sec\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right] =$$

$$= \frac{350\times10^3}{7.42\times10^{-3}}\left[1+\frac{(6.33)\left(\frac{203}{2}\times10^{-3}\right)}{(0.0503)^2}\sec\frac{\pi}{2}\sqrt{\frac{350}{902.6}}\right]$$

$$= 68{,}615{,}044\ \text{N/m}^2 = \boxed{68.62\ \text{MPa}}$$

- An alternate solution for Part (b):



$$y_m = 5\ \text{mm}$$

$$e = 6.33\ \text{mm}$$

$$P = 350\times10^3\ N$$

$$M = P(y_m + e) = 350[0.005+0.00633] = 3.966\ \text{kN-m}$$

$$\sigma_m = \frac{P}{A}+\frac{M}{S_y} = \frac{350}{7.42\times10^{-3}}+\frac{3.966}{184.5\times10^{-6}} = \boxed{68.67\ \text{MPa}}$$

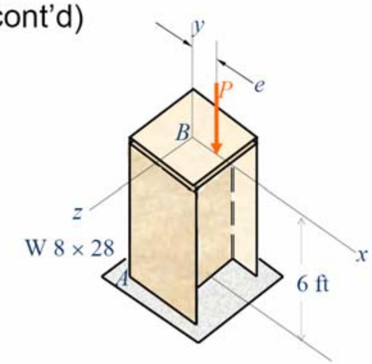
Oops: Collapse of the Second Narrows Bridge, Vancouver

- Example 2

  An axial load P is applied at a point located on the x axis at a distance e = 0.60 in. from the geometric axis of the W8 × 28 rolled-steel column AC (Fig. 9). Knowing that the column is free at its top B and fixed at its base A and that σ_y = 36 ksi and $E = 29 \times 10^6$ psi, determine the allowable load P if a factor of safety of 2.5 with respect to yield is required.

- Example 2 (cont'd)

Figure 9



- For W 8 × 28

$$I_y = 21.7\ \text{in}^4 \qquad A = 8.25\ \text{in}^2$$

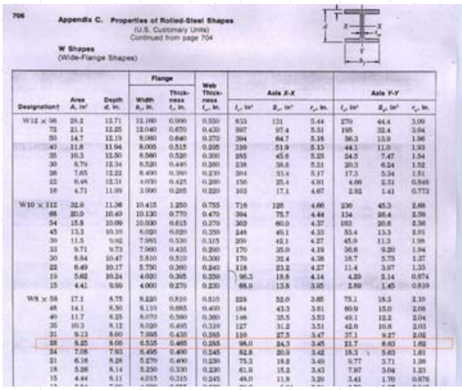
$$c = \frac{6.535}{2}\ \text{in} = 3.2675 \qquad r_y = 1.62\ \text{in}$$

- One-end fixed, one-end free column,

$$L' = 2(6) = 12\ \text{ft}$$

- Therefore

$$\frac{L'}{r_y} = \frac{12 \times 12}{1.62} = 88.89 \quad \text{and} \quad \frac{ec}{r^2} = \frac{0.6(3.2675)}{(1.62)^2} = 0.747$$



706 **Appendix C. Properties of Rolled-Steel Shapes**
(U.S. Customary Units)
Continued from page 704

W Shapes
(Wide-Flange Shapes)

Designation†	Area A, in²	Depth d, in.	Flange Width b_f, in.	Flange Thickness t_f, in.	Web Thickness t_w, in.	Axis X-X I_x, in⁴	Axis X-X S_x, in³	Axis X-X r_x, in.	Axis Y-Y I_y, in⁴	Axis Y-Y S_y, in³	Axis Y-Y r_y, in.
W12 × 96	28.2	12.71	12.160	0.900	0.550	833	131	5.44	270	44.4	3.09
72	21.1	12.25	12.040	0.670	0.430	597	97.4	5.31	195	32.4	3.04
50	14.7	12.19	8.080	0.640	0.370	394	64.7	5.18	56.3	13.9	1.96
40	11.8	11.94	8.005	0.515	0.295	310	51.9	5.13	44.1	11.0	1.93
35	10.3	12.50	6.560	0.520	0.300	285	45.6	5.25	24.5	7.47	1.54
30	8.79	12.34	6.520	0.440	0.260	238	38.6	5.21	20.3	6.24	1.52
26	7.65	12.22	6.490	0.380	0.230	204	33.4	5.17	17.3	5.34	1.51
22	6.48	12.31	4.030	0.425	0.260	156	25.4	4.91	4.66	2.31	0.848
16	4.71	11.99	3.990	0.265	0.220	103	17.1	4.67	2.82	1.41	0.773
W10 × 112	32.9	11.36	10.415	1.250	0.755	716	126	4.66	236	45.3	2.68
68	20.0	10.40	10.130	0.770	0.470	394	75.7	4.44	134	26.4	2.59
54	15.8	10.09	10.030	0.615	0.370	303	60.0	4.37	103	20.6	2.56
45	13.3	10.10	8.020	0.620	0.350	248	49.1	4.32	53.4	13.3	2.01
39	11.5	9.92	7.985	0.530	0.315	209	42.1	4.27	45.0	11.3	1.98
33	9.71	9.73	7.960	0.435	0.290	170	35.0	4.19	36.6	9.20	1.94
30	8.84	10.47	5.810	0.510	0.300	170	32.4	4.38	16.7	5.75	1.37
22	6.49	10.17	5.750	0.360	0.240	118	23.2	4.27	11.4	3.97	1.33
19	5.62	10.24	4.020	0.395	0.250	96.3	18.8	4.14	4.29	2.14	0.874
15	4.41	9.99	4.000	0.270	0.230	68.9	13.8	3.95	2.89	1.45	0.810
W8 × 58	17.1	8.75	8.220	0.810	0.510	228	52.0	3.65	75.1	18.3	2.10
48	14.1	8.50	8.110	0.685	0.400	184	43.3	3.61	60.9	15.0	2.08
40	11.7	8.25	8.070	0.560	0.360	146	35.5	3.53	49.1	12.2	2.04
35	10.3	8.12	8.020	0.495	0.310	127	31.2	3.51	42.6	10.6	2.03
31	9.13	8.00	7.995	0.435	0.285	110	27.5	3.47	37.1	9.27	2.02
28	8.25	8.06	6.535	0.465	0.285	98.0	24.3	3.45	21.7	6.63	1.62
24	7.08	7.93	6.495	0.400	0.245	82.8	20.9	3.42	18.3	5.63	1.61
21	6.16	8.28	5.270	0.400	0.250	75.3	18.2	3.49	9.77	3.71	1.26
18	5.26	8.14	5.250	0.330	0.230	61.9	15.2	3.43	7.97	3.04	1.23
15	4.44	8.11	4.015	0.315	0.245	48.0	11.8	3.29	3.41	1.70	0.876

- Since σ_y = 36 ksi, and $E = 29 \times 10^6$ ksi, Fig. 5 can be used:

  We read

  $$\frac{P}{A} \cong 15; \text{ therefore, } P = 15A = 15(8.25) = 123.75 \text{ kips}$$

- Thus

  $$P_{\text{allowable}} = \frac{P}{\text{FS}} = \frac{123.75}{2.5} = 49.5 \cong \boxed{50 \text{ kips}}$$

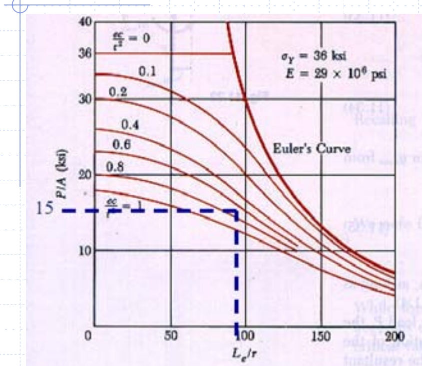


Figure 5.
Load per unit area, *P/A*, causing yield in column (US Customary units)
(Beer & Johnston 1992)

– General Iterative Procedure:

Suppose that we do not have the curves provided in Fig. 5, or we do have the curves but our problem consists of a column that has different material (e.g., σ_y = 50 ksi), how can we evaluate the eccentric load P for Example 2?

A general trial and error (iterative) procedure can be used as follows:

- Using Eqs. 20 or 22, we assume an initial (guess) value for *P* in the right-hand side of the equation; let it be 20 kips, hence

$$P = \frac{\sigma_{\max} A}{1+\frac{ec}{r^2}\sec\left(\frac{1}{2}\sqrt{\frac{P}{EA}}\frac{L'}{r}\right)} = \frac{36(8.25)}{1+0.747\left(\frac{1}{\cos\left[\frac{1}{2}\sqrt{\frac{20}{29\times10^3(8.25)}}(88.89)\right]}\right)} = 163.80 \text{ kips}$$

The revised value *P* = 163.80 kips can now be substituted in the right-hand side of the same equation to produce yet another revised value as follows:

$$P = \frac{\sigma_{\max} A}{1+\frac{ec}{r^2}\sec\left(\frac{1}{2}\sqrt{\frac{P}{EA}}\frac{L'}{r}\right)} = \frac{36(8.25)}{1+0.747\left(\frac{1}{\cos\left[\frac{1}{2}\sqrt{\frac{163.80}{29\times10^3(8.25)}}(88.89)\right]}\right)} = 103.01 \text{ kips}$$

A third iteration using a revised value for *P* = 103.01 kips, gives

$$P = \frac{\sigma_{\max} A}{1+\frac{ec}{r^2}\sec\left(\frac{1}{2}\sqrt{\frac{P}{EA}}\frac{L'}{r}\right)} = \frac{36(8.25)}{1+0.747\left(\frac{1}{\cos\left[\frac{1}{2}\sqrt{\frac{103.01}{29\times10^3(8.25)}}(88.89)\right]}\right)} = 132.79 \text{ kips}$$

- The iterative procedure is continued until the value of the eccentric load P converges to the exact solution of 123.53 kips, as shown in the spreadsheet (Excel) result of Table 1.
- Therefore,

$$P_{\text{allowable}} = \frac{P}{\text{FS}} = \frac{123.53}{2.5} = 49.4 \cong \boxed{50 \text{ kips}}$$

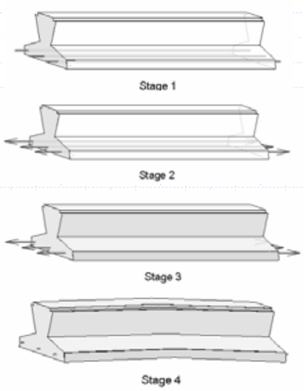
Table 1. Spreadsheet Result

Initial Value of P →

P (kip)	
20.00	123.63
163.79	123.48
103.01	123.55
132.79	123.52
119.10	123.53
125.59	123.53
122.56	123.53
123.98	123.53
123.31	123.53

$$P = \frac{\sigma_{\max} A}{1 + \frac{ec}{r^2} \sec\left(\frac{1}{2}\sqrt{\frac{P}{EA}}\frac{L'}{r}\right)}$$

Pre-stressed Concrete



Stage 1
Tendons and reinforcement are positioned in the beam mould.

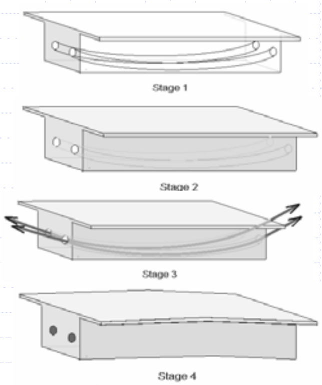
Stage 2
Tendons are stressed to about 70% of their ultimate strength.

Stage 3
Concrete is cast into the beam mould and allowed to cure to the required initial strength.

Stage 4
When the concrete has cured the stressing force is released and the tendons anchor themselves in the concrete.

Prestressed Concrete

Post-tensioned Beams



Stage 1
Cable ducts and reinforcement are positioned in the beam mould. The ducts are usually raised towards the neutral axis at the ends to reduce the eccentricity of the stressing force.

Stage 2
Concrete is cast into the beam mould and allowed to cure to the required initial strength.

Stage 3
Tendons are threaded through the cable ducts and tensioned to about 70% of their ultimate strength.

Stage 4
Wedges are inserted into the end anchorages and the tensioning force on the tendons is released. Grout is then pumped into the ducts to protect the tendons.

Post-tensioned Beams

Loss of Prestress

When the tensioning force is released and the tendons are anchored to the concrete a series of effects result in a loss of stress in the tendons. The effects are :

- relaxation of the steel tendons
- elastic deformation of the concrete
- shrinkage and creep of the concrete
- slip or movement of the tendons at the anchorages during anchoring
- other causes in special circumstances, such as when steam curing is used with pre-tensioning.

Total losses in prestress can amount to about **30%** of the initial tensioning stress.

Arrgggh, it’s over!

Burj Dubai Tower
800 metres (2,640 feet) tall
Upward-spiral design
Exterior cladding system
Built of glass, aluminium, concrete, steel
Designed by Skidmore, Owings & Merrill, Chicago
Total floor space of 500,000 sq m (5.35 million sq ft)

Solution to Problem 3: